Stochastic heat equation with super-linear drift and multiplicative noise on R^d

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- Consider the following stochastic heat equation (SHE)

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x) \qquad (1)$$

for $t \in [0,T]$, $x \in \mathbb{R}^d$.

- Initial condition u_0, a bounded function.
- $\dot{W}$ is a centered Gaussian noise with covariance structure

$$\mathbb{E}\left[\dot{W}(s,y)\dot{W}(t,x)\right] = \delta(t-s)f(x-y)\,,$$

- f is a non-negative, non-negative definite locally integrable function.
- b and σ are locally Lipschitz.
- Question: existence and uniqueness of the solution.

Gaussian noise

- $W = \{W(\varphi), \varphi \in C_0^\infty([0,T] \times \mathbb{R}^d)\}$ is a zero mean Gaussian family with covariance

$$E(W(\varphi)W(\psi)) = \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t,x) f(x-y) \psi(t,y) dx dy dt$$
$$= \int_0^T \langle \varphi(t,\cdot), \psi(t,\cdot) \rangle_{\mathcal{H}} dt$$

- $W(\varphi)$ can be extended to $W(\mathbf{1}_{[0,t]} \mathbf{1}_{[0,x]})$, which is denoted by $W(t,x)$.
- $\dot{W}(t,x) := \frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}$, in the distributional sense.

- The solution is understood in the mild form:

$$u(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) b(u(s,y)) dy ds + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \sigma(u(s,y)) W(ds,dy) .$$

- $p_t(x)$ is the heat kernel,

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} .$$

- The stochastic integral is the Walsh integral.
- Properties of Walsh integral:

$$(1) \quad \mathbb{E} \int_0^t \int_{\mathbb{R}^d} X(s,y) W(ds,dy) = 0$$

$$(2) \quad \mathbb{E} \left(\int_0^t \int_{\mathbb{R}^d} X(s,y) W(ds,dy) \right)^2 = \mathbb{E} \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} X(s,y) X(s,y') f(y-y') dy dy' ds .$$

Classical case

- We temporarily assume that
  - b and σ globally Lipschitz, Lipschitz coefficients L_b and L_σ.
  - $b(0) = \sigma(0) = 0$.
  - u_0 is a bounded function.
- Dalang's condition:

$$\int_{\mathbb{R}^d} \frac{\hat{f}(\xi)}{1 + |\xi|^2} d\xi < \infty .$$

- Existence and uniqueness: Picard iteration.
- Define: $u_1(t,x) = p_t * u_0(x)$, and

$$\begin{aligned} u_{n+1}(t,x) =& p_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) b(u_n(s,y)) dyds \\ &+ \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \sigma(u_n(s,y)) W(ds,dy) \end{aligned}$$

Second moment both sides:

$$
\begin{aligned}
E|u_{n+1}(t,x)|^2 \lesssim & |p_t * u_0(x)|^2 \\
& + \left(L_b \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \|u_n(s,y)\|_{L^2(\Omega)} dyds \right)^2 \\
& + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t-s}(x-y) p_{t-s}(x-y') f(y-y') \\
& \quad \times L_\sigma \|u_n(s,y)\|_{L^2(\Omega)} L_\sigma \|u_n(s,y')\|_{L^2(\Omega)} dydy'ds
\end{aligned}
$$

- Supremum over the spatial variable,

$$\sup_x \mathbb{E}|u_{n+1}(t,x)|^2 \lesssim \sup_x |p_t * u_0(x)|^2 + L_b^2 t \int_0^t \sup_x \mathbb{E}|u_n(s,x)|^2 ds + L_\sigma^2 \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t-s}(y)p_{t-s}(y')f(y-y')dydy' \times \sup_x \mathbb{E}|u_n(s,x)|^2 ds$$

- Dalang's condition $\int_{\mathbb{R}^d} \frac{\hat{f}(\xi)}{1+|\xi|^2} d\xi < \infty \Longleftrightarrow$ integrability of

$$\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t-s}(y)p_{t-s}(y')f(y-y')dydy'ds = \int_0^t \int_{\mathbb{R}^d} e^{-(t-s)|\xi|^2}\hat{f}(\xi)d\xi ds = \int_{\mathbb{R}^d} \frac{1-e^{-t|\xi|^2}}{|\xi|^2}\hat{f}(\xi)d\xi$$

- Existence: replace u_n above by $u_n - u_{n-1}$. We get a contraction, $u_n(t,x)$ is a Cauchy sequence.
- Uniqueness: standard argument.
- We also get the moment growth

$$E|u(t,x)|^p \leq \|u_0\|^p_{L^\infty(\mathbb{R}^d)} e^{tC_{p,f}} ,$$

- If assume the improved Dalang's condition

$$\int_{\mathbb{R}^d} \frac{\hat{f}(\xi)}{(1+|\xi|^2)^{1-\alpha}} d\xi < \infty , \quad \text{for some } 0 < \alpha \leq 1 ,$$

- Moment bounds

$$E|u(t,x)|^p \leq \|u_0\|^p_{L^\infty(\mathbb{R}^d)} \exp\left(Ctp^{1+\frac{1}{\alpha}}\right) .$$

- $u(t,x) \in C^{\alpha/2-,\alpha-}((0,T] \times \mathbb{R}^d)$.
- If b or σ is not globally Lipschitz, Picard iteration does not work.

Back to superlinear problem, $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + b(u) + \sigma(u)\dot{W}$

- Bonder and Groisman 09': additive noise, bounded interval.

$$\text{Osgood condition } \int_1^\infty \frac{du}{b(u)} < \infty \implies \text{finite time blowup.}$$

- Foondun and Nualart 21': additive noise, more general domain.

$$\text{Osgood condition} \iff \text{finite time blowup.}$$

- Salins 21': 1, additive noise, super-linear drift. 2, multiplicative noise, globally Lipschitz drift term, $\mathbb{R}^d$.
- Dalang, Khoshnevisan and Zhang, 19': multiplicative noise, super-linear drift term, space-time white noise, $[0, 1]$.

$$|b(z)| = O(|z| \log |z|) \qquad |\sigma(z)| = o(|z|(\log |z|)^{1/4}), \quad |z| \to \infty$$

- Millet and Sanz-Solé, 21': stochastic wave equation, multiplicative noise, super-linear drift term, 1,2,3-d.

Main ideas in these papers:

- The equation is in a bounded domain $x \in D$.
- Truncate b and σ, $b_n(u) = u$ for $|u| \le n$ and $b_n(u) = b(\pm n)$ for $|u| > n$.
- b_n, σ_n globally Lipschitz $\implies$ unique solution $u_n(t,x)$.
- Control the size of $|u_n(t,x)|$,

$$\tau_n = \inf\left\{ t : \sup_{x \in D} |u_n(t,x)| > n \right\}.$$

- Before τ_n, $b_n(u_n(t,x)) = b(u_n(t,x))$, solution is constructed from 0 to τ_n.
- Show $\tau_n \ge T$ a.s. as $n \to \infty \implies$ global solution.

- Show that $P(\tau_n < T) \to 0$ as $n \to \infty$,

$$\tau_n < T \Longleftrightarrow \sup_{t \le T} \sup_{x \in D} |u_n(t,x)| > n$$

- Chebyshev's inequality.

$$P\{\tau_n < T\} \le \frac{1}{n^p} \mathbb{E} \sup_{t \le T} \sup_{x \in D} |u_n(t,x)|^p$$

- Kolmogorov continuity theorem to estimate

$$\mathbb{E} \sup_{t \le T} \sup_{x \in D} |u_n(t,x)|^p .$$

- Bounded domain is essential.
- Wave equation has finite speed propagation, essentially in a bounded space.
- For stochastic heat equation, $\sup_{x \in \mathbb{R}} u(t,x)$ may be infinity for any $t > 0$.

Theorem (Conus, Joseph, Khoshnevisan 2013)

Let $u_0 > 0$ be uniformly bounded away from 0 and ∞ and u(t,x) satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{W}\,.$$

1. *If $\sigma > 0$ is uniformly bounded away from 0, then a.s.*

$$\limsup_{|x|\to\infty} \frac{u(t,x)}{(\log|x|)^{1/6}} \geq C\,,$$

2. *If $\sigma(u) = u$,*

$$\log \sup_{x\in[-R,R]} u(t,x) \approx (\log R)^{2/3}\,,\ \text{as } R\to\infty\,.$$

- Also for additive noise, the solution is not bounded on $\mathbb{R}^d$.

- Kolmogorov continuity theorem does not work for the whole $\mathbb{R}^d$.
- Still want to estimate

$$\mathbb{E} \sup_{t\le T} \sup_{x\in\mathbb{R}^d} |u_n(t,x)|^p \,.$$

- u_0 has sufficient decay + Da Prato and Zabczyk's factorization method $\Longrightarrow\ \mathbb{E}\sup_{t\le T, x\in\mathbb{R}^d} |u_n(t,x)|^p$.
- Factorization method

$$Z(t,x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\Phi(s,y)W(ds,dy)\,.$$

$$Z(t,x) = \frac{\sin(\beta\pi/2)}{\pi} \int_0^t \int_{\mathbb{R}^d} (t-r)^{-1+\beta/2} p_{t-r}(x-z) Y(r,z)\, dz dr,$$

$$Y(r,z) = \int_0^r \int_{\mathbb{R}^d} (r-s)^{-\beta/2} p_{r-s}(z-y)\Phi(s,y)W(ds,dy)\,,$$

$$\mathbb{E}\left(\sup_{0\le t\le T, x\in\mathbb{R}^d} |Z(t,x)|^k \right) \le C_T \int_0^T dr \int_{\mathbb{R}^d} dz \mathbb{E}\left(|Y(r,z)|^k \right)\,,$$

Theorem (Chen-H' 2023)

If $u_0 \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, *assume improved Dalang's condition with* $0 < \alpha \leq 1$, $b(0) = \sigma(0) = 0$ *and*

$$|b(u)| = o(u \log u)\,, \quad |\sigma(u)| = o(u(\log u)^{\alpha/2})\,, \text{ as } u \to \infty\,,$$

there exists a unique global solution to $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + b(u) + \sigma(u)\dot{W}$.

- More general b and σ?

Osgood type assumptions

$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + b(u) + \sigma(u)\dot{W}$, $\int_{\mathbb{R}^d} \frac{\hat{f}(\xi)d\xi}{(1+|\xi|^2)^{1-\alpha}} < \infty$

1. Both b and σ are locally Lipschitz continuous.
2. $b(0) = \sigma(0) = 0$,
3. There exists a positive, increasing function $h : [0, \infty) \to (0, \infty)$ such that:
   1. For all $u \in \mathbb{R}$, $|b(u)| \leq h(|u|)$.
   2. (Superlinear growth) $u \to \frac{h(u)}{u}$ is non-decreasing on $\mathbb{R}^+$.
   3. (Osgood-type condition)

$$\int_1^\infty \frac{1}{h(u)} du = +\infty \,.$$

   4. For all $u \in \mathbb{R}$, it holds that

$$|\sigma(u)| \leq |u| \left(\frac{h(|u|)}{|u|}\right)^{\alpha/2} \left(\log\left(\frac{h(|u|)}{|u|}\right)\right)^{-1/2} .$$

Theorem (Chen, Foondun, H', Salins)

Assume that $u_0 \in L^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ *for some* $p \geq 2$. *Also assume the improved Dalang's condition and* b, σ *above. Then,*

1. *There exists a unique solution* $u(t,x)$ *to SHE for all* $(t,x) \in (0,\infty) \times \mathbb{R}^d$.
2. *The solution* $u(t,x)$ *is Hölder continuous:* $u \in C^{\alpha/2-,\alpha-}((0,T] \times \mathbb{R}^d)$ *a.s.*

Osgood type conditions covers:

- $b(u) = o(u \log u)$ and $\sigma(u) = o(u(\log u)^{\alpha/2})$ in Chen-H' 2023.
- Spcae-time white noise: $\alpha = 1/2$,

$$b(u) \sim u \log u\,, \quad \sigma(u) \sim u(\log u)^{1/4}(\log\log u)^{-1/2}$$

- Generally $\alpha \in (0,1]$,

$$b(u) \sim u \prod_{k=1}^{K} \log^k u\,, \quad \sigma(u) \sim u(\log^2 u)^{-1/2} \prod_{k=1}^{K} \left(\log^k u\right)^{\alpha/2}$$

Idea of the proof

- Define the cutoff functions for b and σ:

$$b_n(u) := \begin{cases} b(-3^n) & \text{if } u < -3^n \\ b(u) & \text{if } |u| \le 3^n \\ b(3^n) & \text{if } u > 3^n \end{cases} \quad \text{and} \quad \sigma_n(u) = \begin{cases} \sigma(-3^n) & \text{if } u < -3^n \\ \sigma(u) & \text{if } |u| \le 3^n \\ \sigma(3^n) & \text{if } u > 3^n \end{cases}$$

- Consider the equation

$$u_n(t,x) = \int_{\mathbb{R}^d} p_t(x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)b_n(u_n(s,y))dyds$$
$$+ \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma_n(u_n(s,y))W(ds,dy)$$

- Unique global solution $u_n(t,x)$ for each n.

- Define a sequence of stopping times

$$\tau_n = \inf\{t > 0 : \|u_n(t,\cdot)\|_V > 3^n\}\,.$$

- The V-norm

$$\|\cdot\|_V := \max\left(\|\cdot\|_{L^p(\mathbb{R}^d)}, \|\cdot\|_{L^\infty(\mathbb{R}^d)}\right)\,.$$

- Before the stopping time τ_n, the truncation does not take effect.
- Define the local mild solution by setting

$$u(t,x) = u_n(t,x) \quad \text{when} \quad t < \tau_n\,.$$

- Global solution exists if $\tau_n \to \infty$ with probability one.

- Build a deterministic sequence

$$a_n = \min\left\{\frac{\Theta 3^{n+1}}{h(3^{n+1})}, \frac{1}{n}\right\}, \quad \Theta < \frac{1}{3} \text{ small, chosen later}$$

- Osgood type assumption on $h(u)$,

$$\int_1^\infty \frac{1}{h(u)} du = \infty \implies \sum_{n=1}^\infty a_n = \infty\,.$$

- Want to show that there exists a $q > 1$ such that

$$P(\tau_{n+1} - \tau_n < a_n) \leq C n^{-q}\,, \text{for all } n \in \mathbb{N}\,.$$

- Idea: restart the equation at the stopping times τ_n.

- $$\begin{aligned} &u_{n+1}(\tau_n + t, x) \\ =&p_t * u_{n+1}(\tau_n, x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) b_{n+1}(u_{n+1}(\tau_n + s, y)) dy ds \\ &+ \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) \sigma_{n+1}(u_{n+1}(\tau_n + s, y)) W(\tau_n + ds, dy) \\ =&U_{n+1}(t,x) + I_{n+1}(t,x) + Z_{n+1}(t,x)\,. \end{aligned}$$

- $$\{\tau_{n+1} - \tau_n \le a_n\} \subseteq \left\{ \sup_{t \in [0, (\tau_{n+1} - \tau_n) \wedge a_n]} \|Z_{n+1}(t, \cdot)\|_V \ge 3^n \right\}$$

- Chebyshev inequality + factorization method

$$P(\tau_{n+1} - \tau_n < a_n) \le C n^{-q}\,.$$

Optimality of b

Theorem

Assume that b *is nonnegative, convex,* $b(0) = \sigma(0) = 0$ *and* σ *bounded. If* b *satisfies the finite Osgood condition*

$$\int_1^\infty \frac{1}{b(u)} du < \infty\,.$$

Then, for any $p \geq 2$*, there exists some nonnegative initial condition* $u_0(\cdot) \in V$ *such that solutions will explode in finite time with positive probability.*

Idea of the proof: multiply both sides of mild formulation by $p_{1-t}(x)$ and integrate.

$$\int_{\mathbb{R}^d} p_{1-t}(x)u(t,x)dx = \int_{\mathbb{R}^d} u_0(y)p_1(y)dy + \int_0^t \int_{\mathbb{R}^d} p_{1-s}(y)b(u(s,y))dyds + \int_0^t \int_{\mathbb{R}^d} p_{1-s}(y)\sigma(u(s,y))W(ds,dy)\,.$$

- Written as

$$Y_t = Y_0 + D_t + M_t\,.$$

- Let $X_t = \mathbb{E}Y_t$.

$$X_t \geq \int_{\mathbb{R}^d} u_0(y)p_1(y)dy + \int_0^t b(X_s)ds$$

- X_t blows up at time $\frac{1}{2}$ when u_0 is large.

- $$Y_t = Y_{1/2} + D_t^* + M_t^*\,,$$

$$D_t^* = \int_{1/2}^t \int_{\mathbb{R}^d} p_{1-s}(y) b(u(s,y)) dy ds$$
$$M_t^* = \int_{1/2}^t \int_{\mathbb{R}^d} p_{1-s}(y) \sigma(u(s,y)) W(ds,dy)\,.$$

- Jensen's inequality

$$Y_t \geq Y_{1/2} + M_t^* + \int_{1/2}^t b(Y_s) ds$$

- $Y_{1/2} + M_t^*$ is large for all $t \in [0, \frac{1}{2}]$ with positive probability.
- Y_t blows up with positive probability.

References

1. J. Bonder and P. Groisman: Time-space white noise eliminates global solutions in reaction-diffusion equations. Physica D 238 (2009) 209-215.
2. M. Foondun and E. Nualart: The Osgood condition for stochastic partial differential equations. Bernoulli 27(1), 2021, 295-311.
3. M. Salins: Existence and uniqueness for the mild solution of the stochastic heat equation with non-Lipschitz drift on an unbounded spatial domain, Stochastics and Partial Differential Equations: Analysis and Computations 9(3) (2021), pp. 714-745.
4. M. Salins: Existence and uniqueness of global solutions to the stochastic heat equation with super-linear drift on an unbounded spatial domain, Stochastics and Dynamics (2021)

5. R. Dalang, D. Khoshnevisan and T. Zhang: Global solutions to stochastic reaction-diffusion equations with super-linear drift and multiplicative noise. The Annals of Probability 2019, Vol. 47, No. 1, 519-559
6. A. Millet and M. Sanz-Solé: Global solutions to stochastic wave equations with superlinear coefficients. Stochastic Processes and their Applications 139 (2021) 175-211.
7. L. Chen and J. Huang: Superlinear stochastic heat equation on $\mathbb{R}^d$. Proceedings of AMS, 2023.
8. S.Shang and T. Zhang: Global well-posedness to stochastic reaction-diffusion equations on the real line $\mathbb{R}$ with superlinear drifts driven by multiplicative space-time white noise. Electron. J. Probab. 28 (2023),

Thank you.